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Extremal problems for matrix-valued polynomials on the unit circle and applications to multivariate stationary sequences

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Abstract

The paper is devoted to a matrix generalization of a problem studied by Grenander and Rosenblatt (Trans. Amer. Math. Soc. 76 (1954) 112–126) and deals with the computation of the infimum Δ of $\int_{\mathbb{T}} Q^*(z)M(dz)Q(z)$, where M is a non-negative Hermitian matrix-valued Borel measure on the unit circle $\mathbb{T}$ and Q runs through the set of matrix-valued polynomials with prescribed values of some of their derivatives at a finite set $\mathbb{J}$ of complex numbers. Under some additional assumptions on M and $\mathbb{J}$, the value of Δ is computed and the results are applied to linear prediction problems of multivariate weakly stationary random sequences. A related truncated problem is studied and further extremal problems are briefly discussed.
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1. Introduction

In [22] (cf. also [9]) Szegö had studied the infimum of the L^2-norms (with respect to an absolutely continuous non-negative finite Borel measure on the unit circle $\mathbb{T}$ of

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the complex plane $\mathbb{C}$) of polynomials with prescribed value at some $\alpha \in \mathbb{C}$. Grenander and Rosenblatt [8] extended Szegö's results to the case, that at a finite number of points of $\mathbb{C}$ the values of some derivatives of the polynomials are given. At the same time these authors pointed out that their results can be applied to the theory of univariate weakly stationary random sequences. For example, they used their results to compute the linear prediction error of Kolmogorov's prediction problem, which is characterized by the assumption that the whole past of the sequence is known (see [13,14]).

Although the title of the paper [8] indicated that there might be further applications to prediction theory, it seems that this suggestion has been ignored for a long time. The credit of calling the attention of prediction theorists to the paper [8] goes to Pourahmadi [18]. Choosing the restraints of the extremal problem in an appropriate way he computed the linear prediction error of Nakazi's prediction problem [16], where along with the whole past the values of the sequence at the first n positive integers are assumed to be known ($n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers), and of a slight modification of Kolmogorov's prediction problem, where all but one values of the past are assumed to be known. Pourahmadi concluded his paper with outlining some directions of further investigation. One of them is a generalization to the multivariate case, which is the subject of the present paper.

Let $q \in \mathbb{N}$ and denote by $\mathscr{M}_q$ the algebra of $q \times q$-matrices with complex entries and by $\mathscr{M}_q^{\geqslant}$ the subset of non-negative Hermitian $q \times q$-matrices. For an $\mathscr{M}_q^{\geqslant}$-valued Borel measure M on $\mathbb{T}$, one can introduce the right Hilbert $\mathscr{M}_q$-module $L^2(M)$ of (equivalence classes of) Borel measurable $\mathscr{M}_q$-valued functions on $\mathbb{T}$, which are square-integrable with respect to M (see Section 2 for a more precise definition of $L^2(M)$ and for references). Denote by $\langle \cdot, \cdot \rangle$ the $\mathscr{M}_q$-valued inner product of $L^2(M)$. Then the main extremal problem of the present paper consists in computing $\Delta := \inf \langle Q, Q \rangle$, where Q runs through the set of $\mathscr{M}_q$-valued polynomials with prescribed values of some of their derivatives at a finite set $\mathbb{J}$ of complex numbers. The exact formulation of this problem is given in Section 2.

It turns out that there are close relations between the extremal problems we wish to study and the theory of q-variate weakly stationary random sequences ("stationary sequences" for short). On the one hand, we can apply the extremal problems to linear prediction theory. On the other hand, results from stationary sequences are useful in solving extremal problems. For the reader's convenience, some basic facts on stationary sequences are summarized in Section 3.

Section 4 contains some preliminary assertions on the main extremal problem. Using some facts on stationary sequences we show that if $\mathbb{T} \cap \mathbb{J} = \emptyset$, then the singular part of the measure M does not influence Δ. Another result is proved under the assumption that M is absolutely continuous and its Radon–Nikodym derivative has the form $\Phi^*\Phi$ for some $\mathscr{M}_q$-valued outer function Φ of the Hardy class H_q^2. Then from a characterization of $\mathscr{M}_q$-valued outer functions (cf. [17, p. 38]) it can be derived easily that $\Delta = 0$ if $\mathbb{J}$ is outside the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$.

From point of view of applications to prediction theory it is most important to compute Δ if $\mathbb{J}$ is a subset of $\mathbb{D}$. This can be done under some additional assumptions on M. We follow the way of Grenander and Rosenblatt [8] in the univariate case. First we study a truncated extremal problem, where additionally to the restraints of the main extremal problem the degree of Q does not exceed a certain number $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then we let t tend to ∞.

Section 5 is devoted to the truncated extremal problem, which is solved under the additional assumption that the measure M is non-degenerate, i.e. M admits a system of orthonormal $\mathscr{M}_q$-valued polynomials. Moreover, we try to give a description of non-degenerate measures.

Section 6 deals with the limit process $t \to \infty$ and with applications to prediction theory. Our main result is obtained under the assumption that the Radon–Nikodym derivative of the absolutely continuous part of M can be written in the form $\Phi^*\Phi$ for some outer function Φ of H_q^2 such that Φ is invertible. Then Δ can be expressed with the aid of Φ (see Theorem 17). The proof of this result is based on some facts about orthogonal $\mathscr{M}_q$-valued polynomials as Delsarte et al. [3] developed. It would be of interest to compute Δ without assuming that Φ is invertible. Some problems occurring in this case are discussed. For the most part, our applications to prediction theory are analogous to Pourahmadi's results [18] concerning univariate weakly stationary random sequences.

The concluding Section 7 is devoted to a brief discussion of further extremal problems, which are closely related to those of the preceding sections.

2. Formulation of the main extremal problem

Let $q \in \mathbb{N}$. For $A \in \mathscr{M}_q$, denote by A^*, A^+, $\mathscr{R}(A)$, $\operatorname{tr} A$, and $\det A$ the adjoint, Moore–Penrose-inverse, range, trace, and determinant, respectively, of A. The identity matrix of $\mathscr{M}_q$ is denoted by I, whereas the symbol 0 stands for a zero matrix, whose size should be clear from the context. If A is invertible, we write A^{-1} for its inverse. The set $\mathscr{M}_q^{\geqslant}$ will be equipped with Loewner's semi-ordering, i.e. $B \geqslant A$ if and only if $B - A \in \mathscr{M}_q^{\geqslant}$, $A, B \in \mathscr{M}_q^{\geqslant}$. Notions as infimum or minimum of a subset of $\mathscr{M}_q^{\geqslant}$ are to be understood with respect to Loewner's semi-ordering. If $A \in \mathscr{M}_q^{\geqslant}$, the symbol $A^{\frac{1}{2}}$ denotes the unique non-negative Hermitian square root of A and the inequality $A > 0$ means that A is invertible.

Let $\mathfrak{M}$ be a right Hilbert $\mathscr{M}_q$-module with $\mathscr{M}_q$-valued inner product $\langle \cdot, \cdot \rangle_{\mathfrak{M}}$. Throughout the paper, by a submodule of $\mathfrak{M}$ we mean a closed submodule. If $\mathscr{S}$ is a subset of $\mathfrak{M}$, let $\bigvee_{\mathfrak{M}} \mathscr{S}$ be the submodule of $\mathfrak{M}$ spanned by the elements of $\mathscr{S}$. Many geometrical facts of $\mathfrak{M}$ are similar to those of a Hilbert space. For the reader's convenience, we recall some of them. A detailed study of geometrical properties of a Hilbert $\mathscr{M}_q$-module can be found in [7], for applications to prediction theory cf. [24]. If $\mathfrak{M}_1$ is a submodule of $\mathfrak{M}$ and $F \in \mathfrak{M}$, there exists a unique $F_1 \in \mathfrak{M}_1$ such that $\langle F - F_1, F - F_1 \rangle_{\mathfrak{M}}$ is the minimum of the set $\{\langle F - G, F - G \rangle_{\mathfrak{M}} : G \in \mathfrak{M}_1\}$. The matrix $\langle F - F_1, F - F_1 \rangle_{\mathfrak{M}}$ is called the distance matrix of F to $\mathfrak{M}_1$ and F_1 is the orthogonal projection of F onto $\mathfrak{M}_1$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{M}}$. The submodule

$$\mathfrak{M} \ominus \mathfrak{M}_1 := \{H \in \mathfrak{M} : \langle H, G \rangle_{\mathfrak{M}} = 0 \text{ for all } G \in \mathfrak{M}_1\}$$

is called the orthogonal complement of $\mathfrak{M}_1$. We will write $\mathfrak{M} = \mathfrak{M}_1 \oplus (\mathfrak{M} \ominus \mathfrak{M}_1)$. The element $F - F_1$ is equal to the orthogonal projection of F onto $\mathfrak{M} \ominus \mathfrak{M}_1$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{M}}$, hence,

$$\langle F_1, F_1 \rangle_{\mathfrak{M}} = \min\{\langle F - G, F - G \rangle_{\mathfrak{M}} : G \in \mathfrak{M} \ominus \mathfrak{M}_1\}. \tag{1}$$

Let M be an $\mathscr{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$. Then $\tau := \operatorname{tr} M$ is a non-negative finite Borel measure on $\mathbb{T}$ such that M is absolutely continuous with respect to τ. Let $W := \frac{dM}{d\tau}$ be the corresponding Radon–Nikodym derivative. The set $L^2(M)$ of (equivalence classes of) Borel measurable $\mathscr{M}_q$-valued functions on $\mathbb{T}$ for which $\int_{\mathbb{T}} F^*(z) W(z) F(z) \tau(dz)$ exists form a right Hilbert $\mathscr{M}_q$-module with $\mathscr{M}_q$-valued inner product

$$\langle F, G \rangle_{L^2(M)} := \int_{\mathbb{T}} F^*(z) W(z) G(z) \tau(dz), \quad F, G \in L^2(M),$$

and corresponding scalar inner product $\operatorname{tr}\langle F, G \rangle_{L^2(M)}$. Recall that $L^2(M)$ does not change if τ is replaced by any non-negative σ-finite Borel measure on $\mathbb{T}$, with respect to which M is absolutely continuous. For basic facts about $L^2(M)$ we refer to [19].

If $\mathscr{S}$ is a subset of $L^2(M)$, we denote by $\bar{\mathscr{S}}$ its closure with respect to the topology of $L^2(M)$. Moreover, for simplicity we will write $\bigvee \mathscr{S}$ instead of $\bigvee_{L^2(M)} \mathscr{S}$ and $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{L^2(M)}$.

Let $\mathscr{P}$ denote the (right) $\mathscr{M}_q$-module of $\mathscr{M}_q$-valued polynomials. Considering $\mathscr{P}$ as a subset of $L^2(M)$, we can formulate several extremal problems, which are natural generalizations of the scalar case. Our main extremal problem is the following problem (Π).

(Π) Let $\mathbb{J}$ be a finite subset of $\mathbb{C}$ and let $\mathbb{W}$ be a finite subset of $\mathbb{J} \times \mathbb{N}_0$ such that its projection onto $\mathbb{J}$ is equal to $\mathbb{J}$. For $(\alpha, k) \in \mathbb{W}$, let $B_{(\alpha,k)} \in \mathscr{M}_q$. Furthermore, let

$$\mathscr{L} := \{Q \in \mathscr{P} : Q^{(k)}(\alpha) = 0 \text{ for all } (\alpha, k) \in \mathbb{W}\}$$

and

$$\mathscr{L}_B := \{Q \in \mathscr{P} : Q^{(k)}(\alpha) = B_{(\alpha,k)} \text{ for all } (\alpha, k) \in \mathbb{W}\}.$$

Compute the distance matrix Δ of $Q \in \mathscr{L}_B$ to $\overline{\mathscr{L}}$ with respect to $\langle \cdot, \cdot \rangle$.

Note that Δ does not depend on the choice of $Q \in \mathscr{L}_B$ and that

$$\Delta = \inf\{\langle Q_B, Q_B \rangle : Q_B \in \mathscr{L}_B\} \tag{2}$$

according to (1).

Problems of the present type are often formulated in a form similar to (2) (cf. [1,8]). Our approach (Π) has the advantage that it makes use of geometrical properties of $L^2(M)$. For example, from the geometric facts of Hilbert $\mathscr{M}_q$-modules it immediately follows that problem (Π) has a solution, whereas in the case $q>1$ it is not clear a priori whether the infimum on the right-hand side of (2) exists since $\mathscr{M}_q^{\geqslant}$ is not ordered totally.

For future use we introduce the following notation. If $\alpha \in \mathbb{J}$, let

$$n_\alpha := \max\{k: (\alpha, k) \in \mathbb{W}\}. \tag{3}$$

3. Preliminaries from stationary sequences

We recall some basic facts on stationary sequences presenting them in a form which is convenient for our aims. For a comprehensive introduction to this topic see [24] or the monograph [20].

Let $\mathfrak{H}$ be a (right) Hilbert space over $\mathbb{C}$ with inner product $(\cdot,\cdot)_{\mathfrak{H}}$, which is linear with respect to the second component of $(\cdot,\cdot)_{\mathfrak{H}}$. Let $\mathfrak{H}^q_{\text{row}}$ be the q-fold Cartesian product of $\mathfrak{H}$, where the elements of $\mathfrak{H}^q_{\text{row}}$ are written as rows. For $\mathbf{u} := (u_1, \ldots, u_q)$, $\mathbf{v} := (v_1, \ldots, v_q) \in \mathfrak{H}^q_{\text{row}}$, denote by $[\mathbf{u}, \mathbf{v}]$ their Gramian matrix, i.e.

$$[\mathbf{u}, \mathbf{v}] := ((u_j, v_k)_{\mathfrak{H}})_{j=1,\ldots,q}^{k=1,\ldots,q} \in \mathscr{M}_q,$$

where j and k denote the row and column indices, respectively. It is not hard to see that $\mathfrak{H}^q_{\text{row}}$ is a right Hilbert $\mathscr{M}_q$-module with $\mathscr{M}_q$-valued inner product $\langle\cdot,\cdot\rangle_{\mathfrak{H}^q_{\text{row}}} := [\cdot,\cdot]$.

Let $\mathbb{Z}$ denote the additive group of integers. A map

$$\mathbf{x} : \mathbb{Z} \ni n \to \mathbf{x}(n) \in \mathfrak{H}^q_{\text{row}}$$

is called a q-variate weakly stationary random sequence ("stationary sequence" for short) if $[\mathbf{x}(m), \mathbf{x}(n)]$ depends only on the difference $m-n$ and not on m and n separately, $m, n \in \mathbb{Z}$. The covariance matrix function

$$K: K(n) := [\mathbf{x}(n), \mathbf{x}(0)], \quad n \in \mathbb{Z},$$

of $\mathbf{x}$ is a positive semidefinite $\mathscr{M}_q$-valued function and, hence, has an integral representation

$$K(n) = \int_{\mathbb{T}} z^n M(dz), \quad n \in \mathbb{Z}, \tag{4}$$

where M is a unique $\mathscr{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$, the so-called spectral measure of $\mathbf{x}$. On the other hand, if M is an $\mathscr{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$, then there exists a unique (modulo unitary equivalence) stationary sequence $\mathbf{x}$ such that M is the spectral measure of $\mathbf{x}$.

The submodule $\mathfrak{S}(\mathbf{x})$ of $\mathfrak{H}^q_{\text{row}}$ spanned by the values of $\mathbf{x}$ is called the time domain and the Hilbert $\mathscr{M}_q$-module $L^2(M)$ is called the spectral domain of $\mathbf{x}$.

Throughout the paper, by χ we denote the identity function on $\mathbb{T}$, i.e.

$$\chi(z) = z, \quad z \in \mathbb{T}.$$

From (4) we obtain that the map

$$\mathbf{x}(n) \to \chi^{-n} I, \quad z \in \mathbb{T}$$

can be continued to an isometric isomorphism between the time and the spectral domains of $\mathbf{x}$, which we will call Kolmogorov's isomorphism.

A stationary sequence $\mathbf{x}$ is called completely non-deterministic if

$$\mathfrak{S}^{(-\infty)}(\mathbf{x}) := \bigcap_{n \in \mathbb{Z}} \bigvee_{\mathfrak{H}^q_{\mathrm{row}}} \{\mathbf{x}(m) : m \leqslant n\} = \{0\}$$

and it is called deterministic if

$$\bigvee_{\mathfrak{H}^q_{\mathrm{row}}} \{\mathbf{x}(m) : m \leqslant n\} = \mathfrak{S}(\mathbf{x}), \quad n \in \mathbb{Z}.$$

Each stationary sequence $\mathbf{x}$ admits a unique Wold decomposition into a completely non-deterministic stationary sequence $\mathbf{u}$ and a deterministic stationary sequence $\mathbf{v}$ such that for $m, n \in \mathbb{Z}$ we have the following properties:

- $\mathbf{x}(n) = \mathbf{u}(n) + \mathbf{v}(n)$,
- $\mathbf{u}(m)$ is orthogonal to $\mathbf{v}(n)$,
- $\mathfrak{S}(\mathbf{x}) = \mathfrak{S}(\mathbf{u}) \oplus \mathfrak{S}(\mathbf{v})$,
- $\mathbf{v}(n)$ is equal to the orthogonal projection of $\mathbf{x}(n)$ onto $\mathfrak{S}^{(-\infty)}(\mathbf{x})$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{H}^q_{\mathrm{row}}}$,
- $\mathfrak{S}(\mathbf{v}) = \mathfrak{S}^{(-\infty)}(\mathbf{x})$.

There corresponds a decomposition

$$M = M_{\mathbf{u}} + M_{\mathbf{v}} \tag{5}$$

of the spectral measure M of $\mathbf{x}$ into the sum of the spectral measures $M_{\mathbf{u}}$ of $\mathbf{u}$ and $M_{\mathbf{v}}$ of $\mathbf{v}$ to the Wold decomposition of $\mathbf{x}$. The measures $M_{\mathbf{u}}$ and $M_{\mathbf{v}}$ can be obtained in the following way. Let $F_{\mathbf{u}}$ and $F_{\mathbf{v}}$ denote the images of $\mathbf{u}(0)$ and $\mathbf{v}(0)$, respectively, under Kolmogorov's isomorphism. Then

$$dM_{\mathbf{u}} = F_{\mathbf{u}}^* \, dM \, F_{\mathbf{u}} \quad \text{and} \quad dM_{\mathbf{v}} = F_{\mathbf{v}}^* \, dM \, F_{\mathbf{v}}. \tag{6}$$

Moreover, via Kolmogorov's isomorphism the Wold decomposition induces a decomposition of the spectral domain of $\mathbf{x}$. Denote by $\mathfrak{N}_{\mathbf{u}}$ and $\mathfrak{N}_{\mathbf{v}}$ the images of $\mathfrak{S}(\mathbf{u})$ and $\mathfrak{S}(\mathbf{v})$, respectively, under Kolmogorov's isomorphism, i.e.

$$\mathfrak{N}_{\mathbf{u}} = \bigvee \{\chi^n F_{\mathbf{u}} : n \in \mathbb{Z}\} \tag{7}$$

and

$$\mathfrak{N}_{\mathbf{v}} = \bigvee \{\chi^n F_{\mathbf{v}} : n \in \mathbb{Z}\}. \tag{8}$$

Then from the Wold decomposition we obtain the following results:

$$F_{\mathbf{u}} + F_{\mathbf{v}} = I \quad (\text{as elements of } L^2(M)), \tag{9}$$

$$\mathfrak{N}_{\mathbf{u}} \oplus \mathfrak{N}_{\mathbf{v}} = L^2(M), \tag{10}$$

$$\chi\mathfrak{N}_{\mathbf{u}} = \mathfrak{N}_{\mathbf{u}} \quad \text{and} \quad \chi\mathfrak{N}_{\mathbf{v}} = \mathfrak{N}_{\mathbf{v}}, \tag{11}$$

$$\mathfrak{N}_{\mathbf{v}} = \overline{\{F_{\mathbf{v}}Q : \; Q \in \mathscr{P}\}} \tag{12}$$

and

$$\mathfrak{N}_{\mathbf{v}} \subseteq \bar{\mathscr{P}}. \tag{13}$$

The spectral measure of a completely non-deterministic sequence has another characterization. Denote by λ the normalized (i.e. $\lambda(\mathbb{T}) = 1$) linear Lebesgue measure on $\mathbb{T}$. Let H_q^2 be the Hardy class of $\mathscr{M}_q$-valued functions on the unit disk (cf. [17,23]). We consider H_q^2 as a right Hilbert $\mathscr{M}_q$-module with inner product

$$\langle \Phi, \Psi \rangle_{H_q^2} := \int_{\mathbb{T}} \Phi^*(z)\Psi(z)\lambda(dz), \quad \Phi, \Psi \in H_q^2.$$

A stationary sequence $\mathbf{x}$ is completely non-deterministic if and only if its spectral measure M has the form $dM = \Phi^*\Phi\, d\lambda$ for some outer function $\Phi \in H_q^2$. This implies that if $M = M_{\mathrm{a}} + M_{\mathrm{s}}$ is the decomposition of M into its absolutely continuous part M_{a} and its singular part M_{s} and if $dM_{\mathrm{a}} = \Phi^*\Phi\, d\lambda$ for some outer function $\Phi \in H_q^2$, then

$$M_{\mathbf{u}} = M_{\mathrm{a}} \quad \text{and} \quad M_{\mathbf{v}} = M_{\mathrm{s}}.$$

4. Some preliminary results on the main problem

It seems to be difficult to solve problem (Π) in its full generality. However, under additional assumptions on M or $\mathbb{J}$ partial results can be obtained. We start with an assertion which is an easy consequence of a characterization of $\mathscr{M}_q$-valued outer functions.

Theorem 1. *Assume that there exists an outer function $\Phi \in H_q^2$ such that the measure M has the form $dM = \Phi^*\Phi\, d\lambda$. If $\mathbb{J} \cap \mathbb{D} = \emptyset$, then $\Delta = 0$.*

Proof. Since Φ is outer, we have $\mathscr{R}(\Phi) = \mathfrak{R}$ λ-a.e. for some linear subspace $\mathfrak{R}$ of $\mathbb{C}^q$ (cf. [23, Proposition 2.4 of Chapter 5]). Let

$$H_q^2(\mathfrak{R}) := \{\Psi \in H_q^2 : \; \mathscr{R}(\Psi) \subseteq \mathfrak{R} \; \lambda\text{-a.e.}\}.$$

It is easy to see that the map $F \to \Phi F$ establishes an isometric isomorphism between the right Hilbert $\mathscr{M}_q$-modules $L^2(M)$ and $H_q^2(\mathfrak{R})$. Thus, computing Δ of problem (Π) is equivalent to computing the distance matrix of $\Phi Q \in H_q^2(\mathfrak{R})$ to $\Phi\overline{\mathscr{L}}$ with respect to the inner product of H_q^2, $Q \in \mathscr{L}_B$. Let n_α be defined as in (3), $\alpha \in \mathbb{J}$. Then $\prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha + 1}\mathscr{P} \subseteq \mathscr{L}$, hence,

$$\Phi \prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha + 1}\mathscr{P} \subseteq \Phi\mathscr{L}. \tag{14}$$

Since from [17, p. 38] it follows that $\Phi\prod_{\alpha\in\mathbb{J}}(\chi-\alpha)^{n_\alpha+1}$ is an outer function, (14) yields $\Phi\overline{\mathscr{L}}=H^2_q(\mathfrak{R})$. Thus, $\Delta=0$. $\square$

Our next result shows that if $\mathbb{J}$ is outside the unit circle $\mathbb{T}$, the measure $M_{\mathbf{v}}$ does not influence Δ. To prove it, we consider M as the spectral measure of a stationary sequence and use the notations of Section 3. Let

$$\mathscr{L}_{\mathbf{u}}:=\{F_{\mathbf{u}}Q:\ Q\in\mathscr{L}\}\quad\text{and}\quad\mathscr{L}_{\mathbf{v}}:=\{F_{\mathbf{v}}Q:\ Q\in\mathscr{L}\}.$$

Clearly, $\overline{\mathscr{L}_{\mathbf{u}}}\subseteq\mathfrak{R}_{\mathbf{u}}$ and

$$\overline{\mathscr{L}_{\mathbf{v}}}\subseteq\mathfrak{R}_{\mathbf{v}}. \tag{15}$$

Denote by $\mathtt{P}_{\mathbf{u}}$ and $\mathtt{P}_{\mathbf{v}}$ the orthogonal projections in $L^2(M)$ onto $\mathfrak{R}_{\mathbf{u}}$ and $\mathfrak{R}_{\mathbf{v}}$, respectively.

Lemma 2. *For $F\in L^2(M)$,*

$$\mathtt{P}_{\mathbf{u}}F=F_{\mathbf{u}}F\quad\textit{and}\quad\mathtt{P}_{\mathbf{v}}F=F_{\mathbf{v}}F. \tag{16}$$

Moreover,

$$\overline{\mathscr{L}}\subseteq\overline{\mathscr{L}_{\mathbf{u}}}\oplus\overline{\mathscr{L}_{\mathbf{v}}}. \tag{17}$$

Proof. Relations (7), (8), and (10) yield

$$\int_{\mathbb{T}}z^nF^*_{\mathbf{u}}(z)W(z)F_{\mathbf{v}}(z)\tau(dz)=0,\quad n\in\mathbb{Z},$$

which implies that

$$F^*_{\mathbf{u}}WF_{\mathbf{v}}=0\quad\tau\text{-a.e.} \tag{18}$$

From (9) and (18) we get

$$\langle F,F\rangle=\langle F_{\mathbf{u}}F,F_{\mathbf{u}}F\rangle+\langle F_{\mathbf{v}}F,F_{\mathbf{v}}F\rangle,\quad F\in L^2(M).$$

This shows that the maps $F\to F_{\mathbf{u}}F$ and $F\to F_{\mathbf{v}}F$, $F\in L^2(M)$, are contractions in $L^2(M)$. Combining this with the fact that the trigonometric $\mathscr{M}_q$-valued polynomials are dense in $L^2(M)$ and taking into account (7) and (8), we conclude $F_{\mathbf{u}}L^2(M)\subseteq\mathfrak{R}_{\mathbf{u}}$ and $F_{\mathbf{v}}L^2(M)\subseteq\mathfrak{R}_{\mathbf{v}}$, respectively. Then (9) and (10) yield (16) and, hence, (17). $\square$

Our next goals are to show that if the set $\mathbb{T}\cap\mathbb{J}$ is empty, we have equality in (15) and (17).

Lemma 3. *If $\alpha\in\mathbb{C}\backslash\mathbb{T}$, then $(\chi^*-\alpha)^{-1}\mathfrak{R}_{\mathbf{u}}\subseteq\mathfrak{R}_{\mathbf{u}}$ and $(\chi^*-\alpha)^{-1}\mathfrak{R}_{\mathbf{v}}\subseteq\mathfrak{R}_{\mathbf{v}}$.*

Proof. If $\alpha\in\mathbb{C}\backslash(\mathbb{D}\cup\mathbb{T})$, then

$$(\chi^*-\alpha)^{-1}=-\alpha^{-1}\sum_{n=0}^{\infty}(\alpha^{-1}\chi^*)^n,$$

if $\alpha \in \mathbb{D}$, then

$$(\chi^* - \alpha)^{-1} = \chi \sum_{n=0}^{\infty} (\alpha\chi)^n.$$

Thus, the assertion follows from (11). $\square$

Lemma 4. *If $\mathbb{T} \cap \mathbb{J} = \emptyset$, then $\overline{\mathscr{L}_{\mathbf{v}}} = \mathfrak{N}_{\mathbf{v}}$.*

Proof. Let $G \in L^2(M) \ominus \overline{\mathscr{L}_{\mathbf{v}}}$. Let n_α be defined as in (3), $\alpha \in \mathbb{J}$. Since for $Q \in \mathscr{P}$ the polynomial $Q \prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha + 1}$ belongs to $\mathscr{L}$, we obtain

$$0 = \left\langle G, F_{\mathbf{v}} Q \prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha+1} \right\rangle = \left\langle G \prod_{\alpha \in \mathbb{J}} (\chi^* - \alpha^*)^{n_\alpha+1}, F_{\mathbf{v}} Q \right\rangle.$$

This yields

$$G \prod_{\alpha \in \mathbb{J}} (\chi^* - \alpha^*)^{n_\alpha+1} \in L^2(M) \ominus \mathfrak{N}_{\mathbf{v}} = \mathfrak{N}_{\mathbf{u}}$$

by (12) and (10). Then an application of Lemma 3 gives $G \in L^2(M) \ominus \mathfrak{N}_{\mathbf{v}}$. Thus, we have proved $(L^2(M) \ominus \overline{\mathscr{L}_{\mathbf{v}}}) \subseteq (L^2(M) \ominus \mathfrak{N}_{\mathbf{v}})$, which together with (15) yields the assertion. $\square$

Lemma 5. *If $\mathbb{T} \cap \mathbb{J} = \emptyset$, then $\mathfrak{N}_{\mathbf{v}} \subseteq \overline{\mathscr{L}}$.*

Proof. We prove the equivalent assertion $(L^2(M) \ominus \overline{\mathscr{L}}) \subseteq (L^2(M) \ominus \mathfrak{N}_{\mathbf{v}})$. As in the proof of Lemma 4 we have $Q \prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha+1} \in \mathscr{L}$ for $Q \in \mathscr{P}$. Hence, if $G \in L^2(M) \ominus \overline{\mathscr{L}}$, we get

$$0 = \left\langle G, Q \prod_{\alpha \in \mathbb{J}} (\chi - \alpha)^{n_\alpha+1} \right\rangle = \left\langle G \prod_{\alpha \in \mathbb{J}} (\chi^* - \alpha^*)^{n_\alpha+1}, Q \right\rangle,$$

which implies

$$G \prod_{\alpha \in \mathbb{J}} (\chi^* - \alpha^*)^{n_\alpha+1} \in (L^2(M) \ominus \bar{\mathscr{P}}) \subseteq (L^2(M) \ominus \mathfrak{N}_{\mathbf{v}}) = \mathfrak{N}_{\mathbf{u}}$$

by (13) and (10). Applying Lemma 3, we get $G \in (L^2(M) \ominus \mathfrak{N}_{\mathbf{v}})$. $\square$

Lemma 6. *If $\mathbb{T} \cap \mathbb{J} = \emptyset$, then $\overline{\mathscr{L}} = \overline{\mathscr{L}_{\mathbf{u}}} \oplus \overline{\mathscr{L}_{\mathbf{v}}}$.*

Proof. From (15) and Lemma 5 we conclude $\overline{\mathscr{L}_{\mathbf{v}}} \subseteq \overline{\mathscr{L}}$. It follows $\overline{\mathscr{L}_{\mathbf{u}}} \subseteq \overline{\mathscr{L}}$ by (9). Thus, (17) gives the result. $\square$

Now we can prove the assertion mentioned above. Let M be decomposed according to (5) and let

$$\Delta^{(\mathbf{u})} := \inf\{\langle Q_B, Q_B\rangle_{L^2(M_{\mathbf{u}})} : Q_B \in \mathscr{L}_B\}.$$

Theorem 7. *Let M be an $\mathscr{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$. If $\mathbb{T} \cap \mathbb{J} = \emptyset$, then $\Delta = \Delta^{(\mathbf{u})}$.*

Proof. Let $\mathtt{E}$ be the unit operator and $\mathtt{Q}$, $\mathtt{Q}_{\mathbf{u}}$, and $\mathtt{Q}_{\mathbf{v}}$ be the orthogonal projection in $L^2(M)$ onto $\overline{\mathscr{L}}$, $\overline{\mathscr{L}_{\mathbf{u}}}$, and $\overline{\mathscr{L}_{\mathbf{v}}}$, respectively. Then $\mathtt{E} = \mathtt{P}_{\mathbf{u}} + \mathtt{P}_{\mathbf{v}}$ by (10), $\mathtt{Q} = \mathtt{Q}_{\mathbf{u}} + \mathtt{Q}_{\mathbf{v}}$ by Lemma 6, and $\mathtt{P}_{\mathbf{v}} = \mathtt{Q}_{\mathbf{v}}$ by Lemma 4. Consequently, if $Q \in \mathscr{L}_B$, we obtain

$$\Delta = \langle (\mathtt{E} - \mathtt{Q})Q, (\mathtt{E} - \mathtt{Q})Q\rangle = \langle (\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q, (\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q\rangle.$$

Since $\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}} = \mathtt{P}_{\mathbf{u}}(\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})$, from (16) and (6) then follows

$$\begin{aligned}\Delta &= \langle \mathtt{P}_{\mathbf{u}}(\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q, \mathtt{P}_{\mathbf{u}}(\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q\rangle \\ &= \langle F_{\mathbf{u}}(\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q, F_{\mathbf{u}}(\mathtt{P}_{\mathbf{u}} - \mathtt{Q}_{\mathbf{u}})Q\rangle \\ &= \langle Q - \mathtt{Q}_{\mathbf{u}}Q, Q - \mathtt{Q}_{\mathbf{u}}Q\rangle_{L^2(M_{\mathbf{u}})} = \Delta^{(\mathbf{u})}. \qquad \square\end{aligned}$$

Since $M_{\mathbf{u}}$ is the spectral measure of a completely non-deterministic stationary sequence, there exists an outer function $\Phi \in H_q^2$ such that $dM = \Phi^*\Phi\, d\lambda$. Then Theorems 1 and 7 yield the following corollary.

Corollary 8. *Let M be an $\mathscr{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$. If $(\mathbb{T} \cup \mathbb{D}) \cap \mathbb{J} = \emptyset$, then $\Delta = 0$.*

If the set $\mathbb{T} \cap \mathbb{J}$ is not empty, the equality $\Delta = \Delta^{(\mathbf{u})}$ of Theorem 7 is not true in general. For example, let $q := 1$, M be the Dirac measure concentrated at $\alpha \in \mathbb{T}$, $\mathbb{W} := \{(\alpha, 0)\}$, and $B_{(\alpha,0)} := 1$. Then $\Delta = 1$ and $\Delta^{(\mathbf{u})} = 0$.

5. The truncated extremal problem and non-degenerate measures

To obtain further results on problem (Π) we follow the way of Grenander and Rosenblatt [8] in the univariate case. First we study a truncated problem and then approximate the solution of (Π) by the solution of the truncated problem.

For $t \in \mathbb{N}_0$, let $\mathscr{P}_t$ denote the (right) $\mathscr{M}_q$-module of $\mathscr{M}_q$-valued polynomials whose degree does not exceed t. Then the truncated problem (Π_t) is formulated as follows.

(Π_t) Assume that the set $\mathscr{L}_B \cap \mathscr{P}_t$ is not empty and let $Q \in (\mathscr{L}_B \cap \mathscr{P}_t)$. Compute the distance matrix Δ_t of Q to $\mathscr{L} \cap \mathscr{P}_t$ with respect to $\langle \cdot, \cdot \rangle$.

Note that under the assumption $\mathscr{L}_B \cap \mathscr{P}_t \neq \emptyset$, we have analogously to problem (Π) that Δ_t does not depend on the choice of $Q \in (\mathscr{L}_B \cap \mathscr{P}_t)$ and that

$$\Delta_t = \min\{\langle Q_B, Q_B \rangle : \; Q_B \in (\mathscr{L}_B \cap \mathscr{P}_t)\}. \tag{19}$$

Moreover, from Hermite's interpolation theory (cf. [21, Section 10.2]) it follows that $\mathscr{L}_B \cap \mathscr{P}_t$ is not empty if

$$t \geqslant \sum_{\alpha \in \mathbb{J}} (n_\alpha + 1) - 1,$$

n_α being the numbers defined by (3).

Problem (Π_t) will be solved under the additional assumption that the measure M is non-degenerate. We call an $\mathscr{M}_q^{\geqslant}$-valued Borel measure M on $\mathbb{T}$ non-degenerate, if $\langle Q, Q \rangle \neq 0$ for every non-zero $Q \in \mathscr{P}$. Otherwise M is called degenerate. It is well-known (cf. [3]) that M being non-degenerate is equivalent to the fact that there exists a sequence $(\Phi_n)_{n \in \mathbb{N}_0}$ such that $\Phi_n \in \mathscr{P}_n$ and $\langle \Phi_n, \Phi_m \rangle = \delta_{nm} I$, where δ_{nm} denotes the Kronecker symbol, $n, m \in \mathbb{N}_0$. The sequence $(\Phi_n)_{n \in \mathbb{N}_0}$ is called a sequence of right orthonormal $\mathscr{M}_q$-valued polynomials corresponding to M. It is uniquely defined if one additionally requires that the leading coefficients are positive Hermitian. In what follows, $(\Phi_n)_{n \in \mathbb{N}_0}$ stands for an arbitrary sequence of right orthonormal $\mathscr{M}_q$-valued polynomials corresponding to M.

Since several results of matrix theory rest on the assumption of non-degenerateness (cf. [4] for some examples), a description of non-degenerate measures is of interest. The following considerations are devoted to this question. From another point of view non-degenerate measures were described and studied in [6, Section 5].

A number $z \in \mathbb{T}$ is called a mass point of M, if $M(\{z\}) \neq 0$. It is called a point of full growth of M if $M(\mathscr{O}) > 0$ for each open subset $\mathscr{O}$ of $\mathbb{T}$ containing z. If $q = 1$, a point of full growth of M is simply called a point of growth of M.

It is well-known that for $q = 1$, the measure M is non-degenerate if and only if M has infinitely many points of growth (cf. [9, Section 1.11]). On the other hand, it is clear that for $q \in \mathbb{N} \backslash \{1\}$, there exists a non-degenerate measure M such that the set of all points of full growth of M is empty. It might be hoped that for arbitrary $q \in \mathbb{N}$, the measure M is non-degenerate if it has infinitely many points of full growth. But this is not so as the following simple example shows.

Example 9. Let $q := 2$, M be absolutely continuous with respect to λ and

$$\frac{dM}{d\lambda}(z) := \begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix}, \quad z \in \mathbb{T}.$$

Then all points of $\mathbb{T}$ are points of full growth of M. However, for the polynomial Q:

$$Q(z) := \begin{pmatrix} -z & 0 \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{T},$$

we have $\langle Q, Q \rangle = 0$.

Despite its simplicity Example 9 is representative in a sense and shows the way to a certain characterization of non-degenerate measures.

We call a Borel measurable function $S: \mathbb{T} \to \mathscr{M}_q$ a square root of $W = \frac{dM}{d\tau}$ if $S^* S = W$ τ-a.e.

Obviously, M is degenerate if and only if the following condition (C) is satisfied for some and, hence, for any square root S of W.

(C) There exists a non-zero $\mathbb{C}^q$-valued polynomial P such that $SP = 0$ τ-a.e.

Proposition 10. *Assume that* (C) *is satisfied. Then for any square root S of W, there exist a finite set $\mathbb{S}$ of mass points of M and a rational $\mathscr{M}_q$-valued function R such that R is not invertible and $S = SR$ τ-a.e. on $\mathbb{T} \backslash \mathbb{S}$.*

Proof. Let P be a polynomial satisfying (C). Let

$$\mathbb{S}_0 := \{z \in \mathbb{T} : P(z) = 0\}.$$

For $z \in \mathbb{T} \backslash \mathbb{S}_0$, let $\tilde{R}(z)$ be the orthogonal projection in $\mathbb{C}^q$ onto the linear subspace spanned by

$$P(z) =: \begin{pmatrix} p_1(z) \\ \vdots \\ p_q(z) \end{pmatrix}.$$

Then $R := I - \tilde{R}$ is not invertible and has the matrix representation

$$R = I - \left(\sum_{j=1}^{q} |p_j|^2 \right)^{-1} (p_m p_n^*)_{m=1,\dots,q}^{n=1,\dots,q},$$

which shows that R is rational on $\mathbb{T}$. Moreover, if $\mathbb{S}$ denotes the intersection of $\mathbb{S}_0$ with the set of mass points of M, we have $S = SR$ τ-a.e. on $\mathbb{T} \backslash \mathbb{S}$. $\square$

Let us mention the following special cases of Proposition 10.

Corollary 11. *If the non-negative finite Borel measure* $\det W \, d\tau$ *on* $\mathbb{T}$ *is non-degenerate, then M is non-degenerate. In particular, if*

(i) *τ has infinitely many growth points and W is invertible τ-a.e.*

or if

(ii) *τ does not have mass points and W is invertible on a set of positive τ measure,*

then M is non-degenerate.

On the other hand, it is clear that for $q \in \mathbb{N} \backslash \{1\}$, there exists a non-degenerate measure M such that the non-negative finite Borel measure $\det W \, d\tau$ on $\mathbb{T}$ is the zero measure on $\mathbb{T}$ and, hence, degenerate.

It turns out that condition (C) is also necessary for M having the properties of Proposition 10.

Proposition 12. *Let S be an arbitrary square root of W. If there exist a finite set $\mathbb{S}$ of mass points of M, a Borel measurable $\mathcal{M}_q$-valued function F on $\mathbb{T}$, and a rational $\mathcal{M}_q$-valued function R such that R is not invertible and $S = FR$ τ-a.e. on $\mathbb{T}\backslash\mathbb{S}$, then* (C) *is satisfied.*

Proof. If the number of growth points of τ is finite, the result is obvious. Assume now, that τ has infinitely many growth points. Let $\mathbb{T}_R$ be the subset of $\mathbb{T}$ whose elements are not poles of R. Then R is well defined but not invertible on $\mathbb{T}_R$. Hence, if $z \in \mathbb{T}_R$, the homogeneous linear system of equations with coefficient matrix $R(z)$ has a non-zero solution. Since R is rational, there exists a finite subset $\mathbb{V}$ of $\mathbb{T}_R$ such that on $\mathbb{T}_R\backslash\mathbb{V}$ Gauss' algorithm can be applied in the same way, i.e. at each step we can choose the Pivot element at a place independently of $z \in \mathbb{T}_R\backslash\mathbb{V}$. It follows that there exists a non-zero $\mathbb{C}^q$-valued rational function R_1 such that $R(z)R_1(z) = 0$ and, hence, a non-zero $\mathbb{C}^q$-valued polynomial P_1 such that

$$R(z)P_1(z) = 0, \quad z \in \mathbb{T}_R\backslash\mathbb{V}. \tag{20}$$

Let $z_0 \in \mathbb{T}\backslash\mathbb{S}$ be a mass point of M or, equivalently, of τ. Then $S(z_0) = F(z_0)R(z_0)$. In particular, z_0 is not a pole of R. Thus, R is continuous at z_0 and (20) yields $R(z_0)P_1(z_0) = 0$. We obtain $RP_1 = 0$ τ-a.e. on $\mathbb{T}\backslash\mathbb{S}$. It follows that (C) is satisfied if $\mathbb{S}$ is empty. Otherwise, let $\{z_1, \ldots, z_m\}$ be the set $\mathbb{S}$, $m \in \mathbb{N}$. Define $P := P_1 \prod_{n=1}^{m} (\chi - z_n)$. Then $RP = 0$ τ-a.e. $\square$

Now we study problem (Π_t). We solve it under the assumption that the measure M is non-degenerate. To do this we first transmit it to another right Hilbert $\mathcal{M}_q$-module.

Let us order the finite set $\mathbb{W}$ of problem (Π) or (Π_t) totally. If $\mathbb{W}$ or $\mathbb{W} \times \mathbb{W}$ occur as index sets of vectors or matrices, respectively, we assume that their entries are ordered according to the ordering of $\mathbb{W}$. Moreover, by κ denote the number of elements of $\mathbb{W}$.

For $m \in \mathbb{N}$, let $\mathfrak{M}_m := \mathcal{M}_{q,\mathrm{row}}^m$ be the m-fold Cartesian product of $\mathcal{M}_q$, where the elements of $\mathfrak{M}_m$ are written as rows. For each $\mathbf{X} := (X_1, \ldots, X_m), \mathbf{Y} := (Y_1, \ldots, Y_m) \in \mathfrak{M}_m$, and $A \in \mathcal{M}_q$, we define

$$\mathbf{X} + \mathbf{Y} := (X_1 + Y_1, \ldots, X_m + Y_m), \quad \mathbf{X}A := (X_1A, \ldots, X_mA)$$

and

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathfrak{M}_m} := \sum_{j=1}^{m} X_j^* Y_j.$$

This way $\mathfrak{M}_m$ becomes a right Hilbert $\mathcal{M}_q$-module with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{M}_m}$.

Let $(\Phi_n)_{n\in\mathbb{N}_0}$ be a sequence of right orthonormal $\mathcal{M}_q$-valued polynomials corresponding to M. It is not hard to see that the map

$$\mathscr{P}_t \ni Q \to (\langle \Phi_0, Q\rangle, \dots, \langle \Phi_t, Q\rangle) \in \mathfrak{M}_{t+1}$$

establishes an isometric isomorphism between the submodule $\mathscr{P}_t$ of $L^2(M)$ and the right Hilbert $\mathcal{M}_q$-module $\mathfrak{M}_{t+1}$, $t\in\mathbb{N}_0$.

For $(\alpha,k)\in\mathbb{W}$, denote

$$\mathbf{\Phi}_{(\alpha,k)} := ([\Phi_0^{(k)}(\alpha)]^*, \dots, [\Phi_t^{(k)}(\alpha)]^*) \in \mathfrak{M}_{t+1}.$$

Then, in view of (19), problem (Π_t) can be formulated in terms of $\mathfrak{M}_{t+1}$ as the following problem $(\Pi_t^{\#})$.

$(\Pi_t^{\#})$ Compute

$$\Delta_t = \min\{\langle \mathbf{X},\mathbf{X}\rangle_{\mathfrak{M}_{t+1}}\colon\ \mathbf{X}\in\mathfrak{M}_{t+1},\ \langle \mathbf{\Phi}_{(\alpha,k)},\mathbf{X}\rangle_{\mathfrak{M}_{t+1}} = B_{(\alpha,k)},\ (\alpha,k)\in\mathbb{W}\}.$$

Introduce the matrix

$$\mathbf{\Gamma}_t := (\langle \mathbf{\Phi}_{(\alpha,k)}, \mathbf{\Phi}_{(\beta,l)}\rangle_{\mathfrak{M}_{t+1}})_{(\alpha,k)\in\mathbb{W}}^{(\beta,l)\in\mathbb{W}} \in \mathcal{M}_{q\kappa}^{\geqslant}, \quad t\in\mathbb{N}_0,$$

where $(\alpha,k)\in\mathbb{W}$ and $(\beta,l)\in\mathbb{W}$ denote the row and column indices, respectively.

By $\mathbf{B}$ we denote that element of $\mathfrak{M}_\kappa$, whose entries are the matrices $B_{(\alpha,k)}^*$, $(\alpha,k)\in\mathbb{W}$, of problem (Π) or (Π_t).

Theorem 13. *Let $t\in\mathbb{N}_0$ and let M be a non-degenerate $\mathcal{M}_q^{\geqslant}$-valued Borel measure on $\mathbb{T}$. Then the distance matrix Δ_t of problem (Π_t) is equal to $\mathbf{B}\mathbf{\Gamma}_t^{+}\mathbf{B}^*$.*

Proof. Let $\mathbf{X}\in\mathfrak{M}_{t+1}$ be such that $\langle \mathbf{\Phi}_{(\alpha,k)},\mathbf{X}\rangle_{\mathfrak{M}_{t+1}} = B_{(\alpha,k)}$, $(\alpha,k)\in\mathbb{W}$. We compute Δ_t according to problem $(\Pi_t^{\#})$. This is equivalent to computing the distance matrix of $\mathbf{X}$ to the orthogonal complement of the submodule $\mathfrak{S}_{\mathbf{\Phi}}$ of $\mathfrak{M}_{t+1}$ spanned by $\Phi_{(\alpha,k)}$, $(\alpha,k)\in\mathbb{W}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{M}_{t+1}}$. Taking into account (1) we see that this can be done by computing $\langle \mathbf{X}_{\mathbf{\Phi}},\mathbf{X}_{\mathbf{\Phi}}\rangle_{\mathfrak{M}_{t+1}}$, where $\mathbf{X}_{\mathbf{\Phi}}$ denotes the orthogonal projection of $\mathbf{X}$ onto $\mathfrak{S}_{\mathbf{\Phi}}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{M}_{t+1}}$. In particular, there exists $A_{(\beta,l)}$, $(\beta,l)\in\mathbb{W}$, such that

$$\mathbf{X}_{\mathbf{\Phi}} = \sum_{(\beta,l)\in\mathbb{W}} \mathbf{\Phi}_{(\beta,l)}A_{(\beta,l)}.$$

Putting this expression into the equalities

$$\langle \mathbf{\Phi}_{(\alpha,k)},\mathbf{X}_{\mathbf{\Phi}}\rangle_{\mathfrak{M}_{t+1}} = B_{(\alpha,k)}, \quad (\alpha,k)\in\mathbb{W},$$

we get

$$\sum_{(\beta,l)\in\mathbb{W}} \langle \mathbf{\Phi}_{(\alpha,k)},\mathbf{\Phi}_{(\beta,l)}\rangle_{\mathfrak{M}_{t+1}}A_{(\beta,l)} = B_{(\alpha,k)}, \quad (\alpha,k)\in\mathbb{W}.$$

According to the definition of $\mathbf{\Gamma}_t$, this can be written as $\mathbf{A}\mathbf{\Gamma}_t = \mathbf{B}$ or $\mathbf{\Gamma}_t\mathbf{A}^* = \mathbf{B}^*$, where $\mathbf{A}\in\mathfrak{M}_\kappa$ has the block entries $A_{(\alpha,k)}^*$, $(\alpha,k)\in\mathbb{W}$. We can conclude that the columns of

$\mathbf{B}^*$ belong to the range of the linear operator $\mathbf{\Gamma}_t$ in $\mathbb{C}^{q\kappa}$ and this yields

$$\mathbf{\Gamma}_t \mathbf{\Gamma}_t^+ \mathbf{B}^* = \mathbf{B}^*. \tag{21}$$

Let $\tilde{\mathbf{B}} := \mathbf{B}\mathbf{\Gamma}_t^+$ and $\tilde{\mathbf{X}} := \sum_{(\beta,l)\in\mathbb{W}} \mathbf{\Phi}_{(\beta,l)} \tilde{B}_{(\beta,l)}$, where $\tilde{B}^*_{(\beta,l)} \in \mathscr{M}_q$, $(\beta,l)\in\mathbb{W}$, denote the block entries of $\tilde{\mathbf{B}} \in \mathfrak{M}_\kappa$. Then $\tilde{\mathbf{X}} \in \mathfrak{S}_{\mathbf{\Phi}}$ and

$$\langle \mathbf{\Phi}_{(\alpha,k)}, \mathbf{X} - \tilde{\mathbf{X}} \rangle_{\mathfrak{M}_{t+1}} = B_{(\alpha,k)} - \sum_{(\beta,l)\in\mathbb{W}} \langle \mathbf{\Phi}_{(\alpha,k)}, \mathbf{\Phi}_{(\beta,l)} \rangle_{\mathfrak{M}_{t+1}} \tilde{B}_{(\beta,l)}, \tag{22}$$

for each $(\alpha,k)\in\mathbb{W}$. Since $\sum_{(\beta,l)\in\mathbb{W}} \langle \mathbf{\Phi}_{(\alpha,k)}, \mathbf{\Phi}_{(\beta,l)} \rangle_{\mathfrak{M}_{t+1}} \tilde{B}_{(\beta,l)}$ is the block entry at place (α,k) of $\mathbf{\Gamma}_t \tilde{\mathbf{B}}^*$, from the definition of $\tilde{\mathbf{B}}$ and (21) we obtain that the right-hand side of (22) is equal to 0, $(\alpha,k)\in\mathbb{W}$. It follows that $\widetilde{\mathbf{X}}$ is the orthogonal projection of $\mathbf{X}$ onto $\mathfrak{S}_{\mathbf{\Phi}}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{M}_{t+1}}$. This yields

$$\begin{aligned}\Delta_t = \langle \tilde{\mathbf{X}}, \tilde{\mathbf{X}} \rangle_{\mathfrak{M}_{t+1}} &= \sum_{(\alpha,k)\in\mathbb{W}} \sum_{(\beta,l)\in\mathbb{W}} \tilde{B}^*_{(\alpha,k)} \langle \mathbf{\Phi}_{(\alpha,k)}, \mathbf{\Phi}_{(\beta,l)} \rangle_{\mathfrak{M}_{t+1}} \tilde{B}_{(\beta,l)} \\ &= \tilde{\mathbf{B}} \mathbf{\Gamma}_t \tilde{\mathbf{B}}^* = \mathbf{B}\mathbf{\Gamma}_t^+ \mathbf{\Gamma}_t \mathbf{\Gamma}_t^+ \mathbf{B}^* = \mathbf{B}\mathbf{\Gamma}_t^+ \mathbf{B}^*. \qquad \square\end{aligned}$$

Remark 14. For $\alpha\in\mathbb{J}$, let n_α be the numbers of (3). If $t \geqslant \sum_{\alpha\in\mathbb{J}} (n_\alpha + 1) - 1$, the existence of Hermite's interpolation polynomials (cf. [21, Section 10.2]) implies that the elements of $\mathbf{\Phi}_{(\alpha,k)} \in \mathfrak{M}_{t+1}$, $(\alpha,k)\in\mathbb{W}$, are $\mathscr{M}_q$-linear independent and, hence, $\mathbf{\Gamma}_t$ is invertible, i.e. $\mathbf{\Gamma}_t^+ = \mathbf{\Gamma}_t^{-1}$.

Remark 15. Note that the result of Theorem 13 remains true (with the same proof) if we replace the assumption that M is non-degenerate by the weaker assumption that M is non-degenerate of order t, i.e. $\langle Q, Q \rangle \neq 0$ for each non-zero $Q \in \mathscr{P}_t$.

6. The main result and applications to stationary sequences

We study problem (Π) under the assumption that the set $\mathbb{J}$ is a subset of $\mathbb{D}$, which is the most important case from point of view of applications to prediction theory. Our result follows from Theorem 13 by letting t tend to ∞. However, we were able to compute $\lim_{t\to\infty} \Delta_t$ under a rather stringent additional condition only. We will assume that the following condition $(\tilde{\mathrm{C}})$ is satisfied by the $\mathscr{M}_q^{\geqslant}$-valued Borel measure M on $\mathbb{T}$.

$(\tilde{\mathrm{C}})$ The absolutely continuous part M_{a} of M has the form $dM_{\mathrm{a}} = \Phi^*\Phi\, d\lambda$ for some outer function $\Phi \in H_q^2$ such that Φ is invertible λ-a.e.

Note that under condition $(\tilde{\mathrm{C}})$ the measure M is non-degenerate. Then our result is a simple consequence of Theorem 13 and properties of orthonormal $\mathscr{M}_q$-valued polynomials, which are stated in [3].

Lemma 16. *Assume that* $(\tilde{\mathrm{C}})$ *is satisfied. Furthermore, let* $(\Phi_n)_{n\in\mathbb{N}_0}$ *be a sequence of right orthonormal* $\mathcal{M}_q$*-valued polynomials corresponding to* M. *Then for* $(v,w)\in\mathbb{D}^2$,

$$\sum_{n=0}^{\infty} \Phi_n(v)\Phi_n^*(w) = (1-vw^*)^{-1}\Phi^{-1}(v)[\Phi^{-1}(w)]^*. \tag{23}$$

The convergence at the left-hand side of (23) *is uniformly on compact subsets of* $\mathbb{D}^2$.

Proof. The assertions are implicitly contained in [3]. For the reader's convenience, we outline the way to obtain them. For $m\in\mathbb{N}_0$, let A_m be the minimizing $\mathcal{M}_q$-valued polynomial according to [3, Theorem 2]. Then [3, Eq. (51) and the Christoffel–Darboux formula (65)] yield

$$\begin{aligned} &A_m(v)A_m^{-1}(0)A_m^*(w) \\ &\quad = (1-vw^*)\sum_{n=0}^{m} \Phi_n(v)\Phi_n^*(w) + vw^*\Phi_m(v)\Phi_m^*(w), \end{aligned} \tag{24}$$

where $(v,w)\in\mathbb{D}^2$ and $m\in\mathbb{N}_0$. Furthermore, [3, Theorems 17 and 20] imply that

$$\Phi^{-1}(v)[\Phi^{-1}(w)]^* = \lim_{m\to\infty} A_m(v)A_m^{-1}(0)A_m^*(w), \quad (v,w)\in\mathbb{D}^2, \tag{25}$$

and [3, Eq. (80)] gives

$$\lim_{m\to\infty} \Phi_m(v) = 0, \quad v\in\mathbb{D}. \tag{26}$$

Letting m tend to ∞ in (24) and taking into account (25) and (26), we get (23). The uniform convergence on compact subsets of $\mathbb{D}^2$ is a consequence of [3, Eq. (81)]. □

Now we can state our main result, whose proof will be omitted since it follows from Theorem 13 and Lemma 16 similarly to the scalar case, cf. Grenander and Rosenblatt's proof of [8, Theorem 1].

Theorem 17. *Let* M *be such that* $(\tilde{\mathrm{C}})$ *is satisfied. Assume that* $\mathbb{J}\subseteq\mathbb{D}$. *Let*

$$\Gamma_{(\alpha,k),(\beta,l)} := \frac{\partial^k}{\partial v^k}\frac{\partial^l}{\partial w^l}\left((1-vw)^{-1}\Phi^{-1}(v)[\Phi^{-1}(w^*)]^*\right)\Big|_{v=\alpha,w=\beta^*},$$

where $(\alpha,k),(\beta,l)\in\mathbb{W}$,

$$\mathbf{\Gamma} := \left(\Gamma_{(\alpha,k),(\beta,l)}\right)_{(\alpha,k)\in\mathbb{W}}^{(\beta,l)\in\mathbb{W}},$$

and let $\mathbf{B}$ *be the same as in Theorem* 13. *Then the distance matrix* Δ *of problem* (Π) *is equal to* $\mathbf{B}\mathbf{\Gamma}^{-1}\mathbf{B}^*$.

It would be of interest to weaken condition $(\tilde{\mathrm{C}})$ by avoiding the assumption of invertibility of Φ. The following considerations illustrate some of the occurring difficulties.

Let M be a non-degenerate measure of the form $dM = \Phi^*\Phi \, d\lambda$, where $\Phi \in H_q^2$ is an outer function whose range is assumed to be (λ-a.e.) a certain non-trivial linear subspace $\mathfrak{R}$ of $\mathbb{C}^q$. Let $\alpha \in \mathbb{D}$, $\mathbb{W} := \{(\alpha, 0)\}$ and $B_{(\alpha,0)} := I$. Then Theorem 13 yields

$$\Delta_t = \left(\sum_{n=0}^{t} \Phi_n(\alpha)\Phi_n^*(\alpha) \right)^{-1}, \quad t \in \mathbb{N}_0,$$

a well-known result (cf. [1]). On the other hand, Δ can be computed directly. In fact, the proof of Theorem 1 shows that computing Δ is equivalent to computing the distance matrix of $\Phi \in H_q^2(\mathfrak{R})$ to $\Phi\overline{\mathscr{L}}$ with respect to the inner product of H_q^2. We have $Q_1 \in \mathscr{L}$ if and only if $Q_1 = (\chi - \alpha)Q$ for some $Q \in \mathscr{P}$. Thus, $\Phi\overline{\mathscr{L}} = (\chi - \alpha)H_q^2(\mathfrak{R})$. It follows that $\Psi \in H_q^2(\mathfrak{R})$ is orthogonal to $\Phi\overline{\mathscr{L}}$ if and only if

$$\begin{aligned} 0 &= \int_{\mathbb{T}} [(z-\alpha)\Phi(z)Q(z)]^*\Psi(z)\lambda(dz) \\ &= \int_{\mathbb{T}} [\Phi(z)Q(z)]^*[z^*\Psi(0) + z^*(\Psi(z) - \Psi(0)) - \alpha^*\Psi(z)]\lambda(dz) \\ &= \int_{\mathbb{T}} [\Phi(z)Q(z)]^*[z^*\Psi(0) - \alpha^*\Psi(z)]\lambda(dz), \quad Q \in \mathscr{P}. \end{aligned} \tag{27}$$

Let $\Psi(w) = \sum_{n=0}^{\infty} \Psi_n w^n$, $w \in \mathbb{D}$, be the Taylor expansion of Ψ. From (27) we obtain that Ψ is orthogonal to $\Phi\overline{\mathscr{L}}$ if and only if $\Psi_{n+1} - \alpha^*\Psi_n = 0$, $n \in \mathbb{N}_0$, which means that $\Psi = (1 - \alpha^*\chi)^{-1}\Psi_0$. Thus, the orthogonal complement $\mathfrak{N}$ of $\Phi\overline{\mathscr{L}}$ is the submodule spanned by $(1 - \alpha^*\chi)^{-1}P_{\mathfrak{R}}$, where $P_{\mathfrak{R}}$ denotes the orthoprojector in $\mathbb{C}^q$ onto $\mathfrak{R}$. Using Cauchy's formula, we easily obtain that the orthogonal projection $\Phi_{\mathfrak{N}}$ of Φ onto $\mathfrak{N}$ is equal to $(1 - |\alpha|^2)(1 - \alpha^*\chi)^{-1}P_{\mathfrak{R}}\Phi(\alpha)$. Then

$$\Delta = \langle \Phi_{\mathfrak{N}}, \Phi_{\mathfrak{N}} \rangle_{H_q^2} = (1 - |\alpha|^2)\Phi^*(\alpha)\Phi(\alpha)$$

by (1) and Cauchy's formula. Because of $\Delta = \lim_{t\to\infty} \Delta_t$, we get

$$\lim_{t\to\infty} \left(\sum_{n=0}^{t} \Phi_n(\alpha)\Phi_n^*(\alpha) \right)^{-1} = (1 - |\alpha|^2)\Phi^*(\alpha)\Phi(\alpha). \tag{28}$$

Since the kernel of $\Phi(\alpha)$ is a non-trivial subspace of $\mathbb{C}^q$, from (28) it follows that $\sum_{n=0}^{\infty} \Phi_n(\alpha)\Phi_n^*(\alpha)$ diverges on a non-trivial subspace of $\mathbb{C}^q$. This shows that an analogue of (23) with Φ^{-1} replaced by Φ^+ cannot be true. Of course, if one considers matrix partitions according to the orthogonal decomposition of $\mathbb{C}^q$ into the range of $\Phi^*(\alpha)$ and the kernel of $\Phi(\alpha)$, one can obtain a certain convergence result for the left upper corner. But this seems to be not too useful since the function whose value at $w \in \mathbb{D}$ is the orthogonal projection onto $\mathscr{R}(\Phi^*(w))$ is not analytic in general.

Now we will apply Theorem 17 to some linear prediction problems of stationary sequences. For $m \in \mathbb{N}$ and a stationary sequence $\mathbf{x} := (\mathbf{x}(n))_{n\in\mathbb{Z}}$, we consider the following problems:

(x1) Compute the distance matrix $\Delta_{(\mathrm{x1})}$ of $\mathbf{x}(0)$ to $\bigvee_{\mathfrak{S}(\mathbf{x})} \{\mathbf{x}(n) : n < 0\}$.

(x2) Compute the distance matrix $\Delta_{(x2)}$ of $\mathbf{x}(0)$ to $\bigvee_{\mathfrak{S}(\mathbf{x})} \{\mathbf{x}(n) :\ n\leqslant m \text{ and } n\neq 0\}$.
(x3) Compute the distance matrix $\Delta_{(x3)}$ of $\mathbf{x}(0)$ to $\bigvee_{\mathfrak{S}(\mathbf{x})} \{\mathbf{x}(n) :\ n<0 \text{ and } n\neq -m\}$.

Let M denote the spectral measure of $\mathbf{x}$. Using Kolmogorov's isomorphism we can formulate problems (x1)–(x3) as extremal problems of type (Π), with special choice of the restraints. For (x1) this is easy, for (x2) and (x3) this can be done analogously to the univariate case (see [18, Section 2]). We obtain the following problems (Π_{x1}) – (Π_{x3}), which are equivalent to the problems (x1)–(x3), respectively:

(Π_{x1}) Compute $\Delta_{(x1)} = \inf\{\langle Q, Q\rangle :\ Q\in\mathscr{P}, Q(0) = I\}$.
(Π_{x2}) Compute $\Delta_{(x2)} = \inf\{\langle Q, Q\rangle :\ Q\in\mathscr{P}, Q^{(m)}(0) = m!I\}$.
(Π_{x3}) Compute $\Delta_{(x3)} = \inf\{\langle Q, Q\rangle :\ Q\in\mathscr{P}, Q(0) = I, Q^{(m)}(0) = 0\}$.

If we assume that M satisfies condition $(\tilde{\mathrm{C}})$ we can apply Theorem 17 and obtain the following assertions.

Solution of (Π_{x1}) *under* $(\tilde{\mathrm{C}})$: We have $\mathbb{W} = \{(0,0)\}$, $B_{(0,0)} = I$, and $\mathbf{\Gamma} = \Phi^{-1}(0)[\Phi^{-1}(0)]^*$, hence

$$\Delta_{(x1)} = \Phi^*(0)\Phi(0), \tag{29}$$

a result which is well-known.

Solution of (Π_{x2}) *under* $(\tilde{\mathrm{C}})$: We have $\mathbb{W} = \{(0,m)\}$ and $B_{(0,m)} = m!I$. Let

$$\Phi^{-1}(w) = \sum_{j=0}^{\infty} D_j w^j, \quad w\in\mathbb{D},$$

be the Taylor expansion of Φ^{-1}. Then similarly to the case $q = 1$ (cf. [18, pp. 7–8]) we get $\mathbf{\Gamma} = (m!)^2\sum_{j=0}^{m} D_j D_j^*$, which yields

$$\Delta_{(x2)} = \left(\sum_{j=0}^{m} D_j D_j^*\right)^{-1}. \tag{30}$$

Solution of (Π_{x3}) *under* $(\tilde{\mathrm{C}})$: We have $\mathbb{W} = \{(0,0),(0,m)\}$, $B_{(0,0)} = I$, and $B_{(0,m)} = 0$. Similarly to the case $q = 1$ (cf. [18, pp. 7–8]) we obtain

$$\mathbf{\Gamma} = \begin{pmatrix} D_0 D_0^* & m! D_0 D_m^* \\ m! D_m D_0^* & (m!)^2 \sum_{j=0}^{m} D_j D_j^* \end{pmatrix}.$$

From Theorem 17 it follows that $\Delta_{(x3)}$ is equal to the left upper corner of $\mathbf{\Gamma}^{-1}$ which is equal to

$$\Delta_{(x3)} = \left(D_0 D_0^* - D_0 D_m^*\left(\sum_{j=0}^{m} D_j D_j^*\right)^{-1} D_m D_0^*\right)^{-1} \tag{31}$$

by Frobenius' formula.

Problems (x1)–(x3) were first studied for $q=1$. We mention the following papers. Problem (x1) is a part of a classical prediction problem and was solved by Kolmogorov [13,14]. Problem (x2) is called Nakazi's prediction problem and was formulated and solved under additional assumptions on M by Nakazi [16]. Cheng, Miamee, and Pourahmadi solved it completely (see [2, Theorem 4]) as well as problem (x3) (see [2, Theorem 5]). Some of these results were generalized to harmonizable stable sequences (cf. [15] and the references quoted there).

For arbitrary $q\in\mathbb{N}$, problem (x1) was extensively studied (cf. [20,24]). The result (30) was given in [5, formula (5.10)], whereas an expression for $\Delta_{(x3)}$ was not correctly stated in [5]. The corresponding formula (5.16) of [5] becomes correct and then coincides with (31) of the present paper if one replaces B_j by B_j^*, $j=0,\dots,n$, and A_0 by A_0^* there.

In all papers above the result of Theorem 17 was not applied explicitly. For $q=1$, Grenander and Rosenblatt [8] pointed out its usefulness to prediction theory. Pourahmadi [18] applied it to the univariate versions of problems (x2) and (x3) and suggested to study the multivariate case along these lines.

7. Associated extremal problems

We briefly mention two further types of extremal problems, which are closely related to problems studied in the previous sections.

First consider the following problems (Π') and (Π'_t), whose solutions can be easily expressed by the solutions of (Π) and (Π_t), respectively, $t\in\mathbb{N}_0$. Let $\mathcal{M}'_q := \{A\in\mathcal{M}_q:\ \det A=1\}$. Let $\mathbb{W}$ and $B_{(\alpha,k)}$, $(\alpha,k)\in\mathbb{W}$, be such as in the formulation of problem (Π) and let

$$\begin{aligned}\mathcal{L}'_B &:= \mathcal{L}_B\mathcal{M}'_q\\ &= \{Q\in\mathcal{P}:\ Q^{(k)}(\alpha)=B_{(\alpha,k)}A, (\alpha,k)\in\mathbb{W},\ \text{for some } A\in\mathcal{M}'_q\}.\end{aligned}$$

(Π') Compute $\delta := \inf\{\operatorname{tr}\langle Q_B,Q_B\rangle:\ Q_B\in\mathcal{L}'_B\}$.

(Π'_t) Assume that $t\in\mathbb{N}_0$ is such that the set $\mathcal{L}'_B\cap\mathcal{P}_t$ is not empty. Compute $\delta_t := \min\{\operatorname{tr}\langle Q_B,Q_B\rangle:\ Q_B\in\mathcal{L}'_B\cap\mathcal{P}_t\}$.

To express δ by Δ, note that $\mathcal{M}'_q\cap\mathcal{M}_q^{\geqslant}=\{AA^*:\ A\in\mathcal{M}'_q\}$ and recall that if $A\in\mathcal{M}_q^{\geqslant}$, then

$$(\det A)^{\frac{1}{q}} = \tfrac{1}{q}\cdot\inf\{\operatorname{tr}(AC):\ C\in\mathcal{M}'_q\cap\mathcal{M}_q^{\geqslant}\}$$

(cf. [11, Problem 19 of Section 7.8]). Thus, we have

$$\begin{aligned}\delta &= \inf\{\operatorname{tr}\langle Q_B,Q_B\rangle:\ Q_B\in\mathcal{L}'_B\}\\ &= \inf\{\operatorname{tr}\langle Q_BA,Q_BA\rangle:\ Q_B\in\mathcal{L}_B, A\in\mathcal{M}'_q\}\\ &= \inf\{\operatorname{tr}(A^*\langle Q_B,Q_B\rangle A):\ Q_B\in\mathcal{L}_B, A\in\mathcal{M}'_q\}\end{aligned}$$

$$=\inf\{\mathrm{tr}(\langle Q_B, Q_B\rangle AA^*):\ Q_B\in\mathscr{L}_B, A\in\mathscr{M}'_q\}$$
$$=\inf\{\mathrm{tr}(\langle Q_B, Q_B\rangle C):\ Q_B\in\mathscr{L}_B, C\in\mathscr{M}'_q\cap\mathscr{M}_q^{\geqslant}\}$$
$$=\inf\{q(\det\langle Q_B, Q_B\rangle)^{\frac{1}{q}}:\ Q_B\in\mathscr{L}_B\} = q(\det\Delta)^{\frac{1}{q}}. \tag{32}$$

Similarly, $\delta_t = q(\det\Delta_t)^{\frac{1}{q}}$, $t\in\mathbb{N}_0$.

As a special case we mention the following result. Let $\mathbb{W} := \{(0,0)\}$, $B_{(0,0)} := I$, and assume that M satisfies $(\tilde{\mathrm{C}})$. Then from (29) and (32) we get

$$\delta = q|\det\Phi(0)|^{\frac{2}{q}},$$

a well-known result announced by Zasukhin [25]. For its proof see [10]. Compare also [5, proof of Theorem 4.4] for an alternative proof and [12] for an L^p-version, $p\in(0,\infty)$, of this result.

Our second remark deals with the fact that all extremal problems of the present paper have "left versions", i.e. one can consider $\mathscr{P}$ as a subset of the left Hilbert $\mathscr{M}_q$-module $L^2_l(M)$ of all (equivalence classes of) Borel measurable $\mathscr{M}_q$-valued functions F on $\mathbb{T}$, such that $\int_{\mathbb{T}} F(z)W(z)F^*(z)\tau(dz)$ exists and study the extremal problems with respect to the inner product

$$\langle F, G\rangle_{L^2_l(M)} := \int_{\mathbb{T}} F(z)W(z)G^*(z)\tau(dz), \quad F, G\in L^2_l(M).$$

Corresponding results can be proved by an adaption of the arguments for the "right" results or by application of this "right" results to the transpose measure $M^\top$. We omit the details. However, there is another method to relate truncated "left" and "right" problems with concrete restraints to truncated "right" and "left" problems, respectively, with restraints which differ from the initial restraints in general. In fact, for $t\in\mathbb{N}_0$ and $Q\in\mathscr{P}_t$, set $\tilde{Q}(z) := z^t Q(\frac{1}{z^*})^*$, $z\in\mathbb{T}$. The map $Q\to\tilde{Q}$ establishes a one-to-one correspondence on $\mathscr{P}_t$, and one has $\langle Q, Q\rangle = \langle\tilde{Q}, \tilde{Q}\rangle_{L^2_l(M)}$. For example, if we set $\mathbb{W} := \{(0,0)\}$ and $B_{(0,0)} := I$, we get

$$\Delta_t = \min\{\langle Q, Q\rangle:\ Q\in\mathscr{P}_t,\ Q(0) = I\}$$
$$= \min\{\langle Q, Q\rangle_{L^2_l(M)}:\ Q\in\mathscr{P}_t,\ Q^{(t)}(0) = t!I\}, \quad t\in\mathbb{N}_0,$$

and if we set $\mathbb{W} := \{(0,t)\}$ and $B_{(0,t)} := t!I$, we get

$$\Delta_t = \min\{\langle Q, Q\rangle:\ Q\in\mathscr{P}_t,\ Q^{(t)}(0) = t!I\}$$
$$= \min\{\langle Q, Q\rangle_{L^2_l(M)}:\ Q\in\mathscr{P}_t,\ Q(0) = I\}, \quad t\in\mathbb{N}_0,$$

this way.

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